Elementary proof of Rayleigh formula for graphs

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Abstract

The Rayleigh monotonicity is a principle from the theory of electrical networks. Its combinatorial interpretation says for each two edges of a graph G, that the presence of one of them in a random spanning tree of G is negatively correlated with the presence of the other edge. In this paper we give a self-contained (inductive) proof of Rayleigh monotonicity for graphs.

Rayleigh monotonicity refers to an intuitive principle in the theory of electrical networks: the total resistance between two nodes in the network does not decrease when we increase the resistance of one branch.

We will refer to a *graph* for what is sometimes called a multigraph in literature, i.e., two vertices may be connected with several edges. When we speak about a subgraph of a graph, we refer only to its edge set; the subgraph is always spanning.

The network can be viewed as a graph whose vertices are nodes and edges are the branches of the network. The graph is weighted, each edge has weight equal to the reciprocal of the resistance of the respective branch.

Let $G = (V, E, w)$ be a graph with weighted edges where $w : E \to \mathbb{R}^+$ is its weight function. For $I \subseteq E$ we define the weight of I by $w(I) = \prod_{e \in I} w(e)$ and for a family $\mathcal{F}$ of sets of edges, $\mathcal{F} \subseteq 2^E$, we define its weight $\|\mathcal{F}\| = \sum_{I \in \mathcal{F}} w(I)$.

We will use $\mathcal{T}_{e_1,e_2}$ to denote the family of spanning trees of G that contain edges e_1 and e_2. Similarly, $\mathcal{T}_{e_1,\overline{e_2}}$, $\mathcal{T}_{\overline{e_1},e_2}$ and $\mathcal{T}_{\overline{e_1},\overline{e_2}}$ denote the families of spanning trees containing the edges without a bar and not containing the edges with a bar.

In 1847, Kirchhoff showed [4] that the resistance between the end-vertices of an edge e_1 of the network is equal to $\frac{1}{w(e_1)}\|\mathcal{T}_{e_1}\|/\|\mathcal{T}\|$, where $\mathcal{T}$ is the family of all spanning trees of G and $\mathcal{T}_{e_1}$ is the family of spanning trees containing e_1. Rayleigh monotonicity principle implies that contracting an edge e_2 does not increase the resistance between the end-vertices of e_1. Therefore

$$\frac{\|\mathcal{T}_{e_1}\|}{\|\mathcal{T}\|} \geq \frac{\|\mathcal{T}_{e_1,e_2}\|}{\|\mathcal{T}_{e_2}\|},$$

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which is equivalent to Theorem 1. Recently, Youngbin Choe [1] found a combinatorial proof of the theorem; the proof uses Jacobi Identity and All Minors Matrix-Tree Theorem. In this paper we give a self-contained combinatorial proof.

Theorem 1.

$$\|\mathcal{T}_{e_1,\overline{e_2}}\|\|\mathcal{T}_{\overline{e_1},e_2}\| \geq \|\mathcal{T}_{e_1,e_2}\|\|\mathcal{T}_{\overline{e_1},\overline{e_2}}\| \tag{1}$$

Proof. Fix an orientation of e_1 and e_2. A subforest F of G is *important* if both $F \cup e_1$ and $F \cup e_2$ form a spanning tree of G. Let C be the unique cycle in $F \cup e_1 \cup e_2$. The cycle C contains both e_1 and e_2. We say that F has *positive orientation* if e_1 and e_2 are consistently oriented along C. Otherwise we say that F has *negative orientation.* Let $\mathcal{C}_+$ and $\mathcal{C}_-$ be the set of all important forests that have positive and negative orientation, respectively.

The statement will be proven by showing that

$$\|\mathcal{T}_{e_1,\overline{e_2}}\|\|\mathcal{T}_{\overline{e_1},e_2}\| - \|\mathcal{T}_{e_1,e_2}\|\|\mathcal{T}_{\overline{e_1},\overline{e_2}}\| = w(e_1)w(e_2)\left(\|\mathcal{C}_+\| - \|\mathcal{C}_-\|\right)^2,$$

or equivalently,

$$\|\mathcal{T}_{e_1,\overline{e_2}}\|\|\mathcal{T}_{\overline{e_1},e_2}\| + 2w(e_1)w(e_2)\|\mathcal{C}_+\|\|\mathcal{C}_-\| = \|\mathcal{T}_{e_1,e_2}\|\|\mathcal{T}_{\overline{e_1},\overline{e_2}}\| + w(e_1)w(e_2)\left(\|\mathcal{C}_+\|^2 + \|\mathcal{C}_-\|^2\right). \tag{2}$$

Equation (2) can be viewed as an equality of two polynomials in variables $w(e)$, $e \in E$. In order to prove it, we shall check that the coefficient of every monomial is the same on the both sides. The *multiplicity* of edge e in monomial $c\prod_{f\in E} w(f)^{\alpha_f}$ is the number α_e. An edge e is *present* in monomial $c\prod_f w(f)^{\alpha_f}$ if its multiplicity is at least one. An edge e is *plentiful* in monomial $c\prod_f w(f)^{\alpha_f}$ if its multiplicity is at least two. The *degree* $d(v)$ of a vertex $v \in V$ is defined by $d(v) = \sum_{f \text{ incident to } v} \alpha_f$. For every monomial $M = \prod_f w(f)^{\alpha_f}$ that is contained (with nonzero coefficient) on one side of (2), it holds $\alpha_{e_1} = \alpha_{e_2} = 1$, $\sum_f \alpha_f = 2(|V|-1)$, $0 \leq \alpha_f \leq 2$ for every $f \in E$ and $d(v) > 1$ for every $v \in V$. Let M be any such a monomial. We have to check that

$$A_{e_1:e_2} + 2A_{+-} = A_{e_1e_2:\emptyset} + A_{++} + A_{--}, \tag{3}$$

where

$$\begin{aligned}
A_{e_1:e_2} &= \#\left\{(T_1,T_2) \mid T_1 \in \mathcal{T}_{e_1,\overline{e_2}}, T_2 \in \mathcal{T}_{\overline{e_1},e_2}, w(T_1)w(T_2) = M\right\},\\
A_{e_1e_2:\emptyset} &= \#\left\{(T_1,T_2) \mid T_1 \in \mathcal{T}_{e_1,e_2}, T_2 \in \mathcal{T}_{\overline{e_1},\overline{e_2}}, w(T_1)w(T_2) = M\right\},\\
A_{+-} &= \#\left\{(F_1,F_2) \mid F_1 \in \mathcal{C}_+, F_2 \in \mathcal{C}_-, w(e_1)w(e_2)w(F_1)w(F_2) = M\right\},\\
A_{++} &= \#\left\{(F_1,F_2) \mid F_1, F_2 \in \mathcal{C}_+, w(e_1)w(e_2)w(F_1)w(F_2) = M\right\},\\
A_{--} &= \#\left\{(F_1,F_2) \mid F_1, F_2 \in \mathcal{C}_-, w(e_1)w(e_2)w(F_1)w(F_2) = M\right\}.
\end{aligned}$$

We will prove Equation (3) by induction on the number of vertices of G. First, we should check that Equation (3) holds for all graphs G with at most 3 vertices. This can be easily done. (Note, that there are infinitely many graphs with at most 3 vertices since multiedges are allowed. This is not a problem as we can without loss of generality assume that G contains only edges present in M.)

Assume that $|V| = n > 3$ and Equation (3) holds for every weighted graph $G' = (V', E', w')$, $|V'| < n$, any choice of edges $e_1', e_2' \in E'$ and any feasible monomial M'.

Every time we use the induction hypothesis, our graph G' will live on a proper subset of vertices of the graph G; edges e_1', e_2' will be the same as in the induction step, i.e., $e_1' = e_1, e_2' = e_2$, unless stated otherwise.

We may assume that the graph G is loopless; we leave out the loops because they do not change any of the terms in (3). Since $\sum_f \alpha_f = 2(|V| - 1)$, there is a vertex v such that $d(v) \leq 3$. Moreover, we can choose v such, that $d(v) \in \{2, 3\}$ and if $d(v) = 3$ then v is incident to at most one of e_1 and e_2. We distinguish two cases.

(i) $d(v) = 2$.
Then either v is incident to two edges (let us call them h, i) present in M or to one plentiful edge h (then we set $i = h$). Recall that if h is plentiful then $h \neq e_1, e_2$.

(a) The edges e_1, e_2 do not coincide with h, i.
Let $M' = M/(w(h)w(i))$, $G' = G - v$. From the induction hypothesis we know that $A'_{e_1:e_2} + 2A'_{+-} = A'_{e_1e_2:\emptyset} + A'_{++} + A'_{--}$. We can add arbitrary one of h and i to every spanning tree of G' and every spanning tree of G has at least one of h and i. Therefore $A_{e_1:e_2} = 2A'_{e_1:e_2}$, $A_{e_1e_2:\emptyset} = 2A'_{e_1e_2:\emptyset}$, $A_{e_1:e_2} = 2A'_{e_1:e_2}$, $A_{+-} = 2A'_{+-}$, $A_{++} = 2A'_{++}$, $A_{--} = 2A'_{--}$ and the statement follows.

(b) One of the edges e_1, e_2 coincides with h, i.
Without loss of generality, let $e_1 = h$. Every important forest contains the edge i, so if F_1 and F_2 are important forests, then $w(e_1)w(e_2)w(F_1)w(F_2) \neq M$. This implies that $A_{+-} = A_{++} = A_{--} = 0$. The mapping

$$(T_1, T_2) \mapsto (T_1 \triangle \{h, i\}, T_2 \triangle \{h, i\})$$

is a bijection between partitions counted in $A_{e_1:e_2}$ and in $A_{e_1e_2:\emptyset}$ and thus Equation (3) holds.

(c) Edges e_1 and e_2 are exactly h and i.
Depending on the orientation of e_1 and e_2, one of the sets $\mathcal{C}_+$, $\mathcal{C}_-$ is empty. Assume, that $\mathcal{C}_- = \emptyset$. Then we have $A_{+-} = A_{--} = 0$. There cannot exist a partition (T_1, T_2) that would be counted in $A_{e_1e_2:\emptyset}$; the edge set of T_2 would not span the vertex v. Thus $A_{e_1e_2:\emptyset} = 0$. The mapping

$$(F_1, F_2) \mapsto (F_1 \cup e_1, F_2 \cup e_2)$$

is a bijection of partitions counted in A_{++} and $A_{e_1:e_2}$. This proves the statement.

(ii) $d(v) = 3$.
Then either v is incident to three edges i, j, h present in M or to one edge j and one plentiful edge h (then we set $i = h$).

(a) None of the edges i, j, h coincides with e_1, e_2.
We can write $A_{e_1:e_2} = A^{i:j,h}_{e_1:e_2} + A^{j:i,h}_{e_1:e_2} + A^{h:i,j}_{e_1:e_2}$, where $A^{X:Y}_{e_1:e_2}$ is defined for two edge sets X and Y as

$$\begin{aligned} A^{X:Y}_{e_1:e_2} = \;& \#\{(T_1, T_2) \mid T_1 \in \mathcal{T}_{e_1,\overline{e_2}}, T_2 \in \mathcal{T}_{\overline{e_1},e_2}, X \subseteq T_1, Y \subseteq T_2, w(T_1)w(T_2) = M\} + \\ & \#\{(T_1, T_2) \mid T_1 \in \mathcal{T}_{e_1,\overline{e_2}}, T_2 \in \mathcal{T}_{\overline{e_1},e_2}, Y \subseteq T_1, X \subseteq T_2, w(T_1)w(T_2) = M\}. \end{aligned}$$

For the numbers $A_{e_1e_2:\emptyset}, A_{+-}, A_{++}$ and A_{--} we define $A^{X:Y}_{e_1e_2:\emptyset}, A^{X:Y}_{+-}, A^{X:Y}_{++}$ and $A^{X:Y}_{--}$ in a similar fashion. We shall show that

$$A^{X:Y}_{e_1:e_2} + 2A^{X:Y}_{+-} = A^{X:Y}_{e_1e_2:\emptyset} + A^{X:Y}_{++} + A^{X:Y}_{--} \tag{4}$$

for $X = \{i\}$, $Y = \{j,h\}$. Then, by symmetry, analogous equalities for $X = \{j\}$, $Y = \{i,h\}$ and $X = \{h\}$, $Y = \{i,j\}$ also hold. Summing them up, we get the statement.

The end-vertices of i and j different from v will be denoted by x and y, respectively. Let $M' = \frac{w(k)}{w(i)w(j)w(h)}M$, $G' = G - v + k$, where $k = xy$ is a new edge connecting vertices x and y (xy may be a multiedge now). We have from the induction hypothesis $A'_{e_1:e_2} + 2A'_{+-} = A'_{e_1e_2:\emptyset} + A'_{++} + A'_{--}$. It is easy to see that $A'_{e_1:e_2} = A^{i:j,h}_{e_1:e_2}$, $A'_{e_1e_2:\emptyset} = A^{i:j,h}_{e_1e_2:\emptyset}$, $A'_{+-} = A^{i:j,h}_{+-}$, $A'_{+-} = A^{i:j,h}_{++}$, $A'_{--} = A^{i:j,h}_{+-}$ and thus (4) holds.

(b) One of the edges e_1, e_2 coincides with i, j, h.
Without loss of generality, assume that $h = e_1$. Let $h = vu, i = vx, j = vy, e_2 = ab$ (with orientation $\overrightarrow{e_2} = \overrightarrow{ba}$). Let $G' = G - v + k_1$, $M' = \frac{w(k_1)}{w(i)w(j)w(k)}M$, $e'_1 = k_1$, $e'_2 = e_2$, where $k_1 = xy$, $G'' = G - v + k_2$, $M'' = \frac{w(k_2)}{w(i)w(j)w(k)}M$, $e''_1 = k_2$, $e''_2 = e_2$, where $k_2 = ux$; $G''' = G - v + k_3$, $M''' = \frac{w(k_3)}{w(i)w(j)w(k)}M$, $e'''_1 = k_3$, $e'''_2 = e_2$, where $k_1 = uy$. We will use induction hypothesis for polynomials M', M'' and M'''. For the edges e'_1, e''_1 and e'''_1 fix orientations $\overrightarrow{e'_1} = \overrightarrow{xy}, \overrightarrow{e''_1} = \overrightarrow{xu}, \overrightarrow{e'''_1} = \overrightarrow{yu}$. Refix[1] orientation of e_1, $\overrightarrow{e_1} = \overrightarrow{vu}$. Then

$$A_{e_1:e_2} = A''_{e''_1:e''_2} + A'''_{e'''_1:e'''_2} + A'_{e'_1e'_2:\emptyset}, \tag{5}$$
$$A_{e_1e_2:\emptyset} = A''_{e''_1e''_2:\emptyset} + A'''_{e'''_1e'''_2:\emptyset} + A'_{e'_1:e'_2}. \tag{6}$$

We shall prove combinatorially that

$$\begin{aligned} A_{++} + A_{--} - 2A_{+-} = {} & \\ = A''_{++} + A''_{--} - 2A''_{+-} \ + \ A'''_{++} + A'''_{--} - 2A'''_{+-} \ - \ A'_{++} - A'_{--} + 2A'_{+-}. & \end{aligned} \tag{7}$$

In order to do so, we will view $2A_{+-}$ as

$$2A_{+-} = \#\{(F_1, F_2) | F_l \in \mathcal{C}_+, F_{3-l} \in \mathcal{C}_-, w(e_1)w(e_2)w(F_1)w(F_2) = M, l \in \{1,2\}\}$$

(and similarly we treat with $2A'_{+-}$, $2A''_{+-}$ and $2A'''_{+-}$). Let $(F^\diamondsuit_1, F^\diamondsuit_2)$ be any partition that is counted in A'_{++}, A'_{--}, $2A'_{+-}$, $\ldots$, $2A'''_{+-}$. Each of $F^\diamondsuit_1$ and $F^\diamondsuit_2$ is a spanning forest of $G - v$ such, that adding the edge e_2 creates a spanning tree of $G - v$. Vertices a and b must be contained in distinct components of $F^\diamondsuit_l$ ($l = 1, 2$). Moreover, no component can contain all the three vertices x, y, u.

Take any partition (F_1, F_2) that is counted in A_{++}, A_{--} or $2A_{+-}$ and delete from it the edges i and j, $F^\heartsuit_l = F_l - \{i,j\}$. It is immediate to see that

[1]Refixing the orientation will not change the validity of the Equation 2, since $A_{+-} = \overline{A}_{+-}, A_{++} = \overline{A}_{--}, A_{--} = \overline{A}_{+}+, A_{e_1:e_2} = \overline{A}_{e_1:e_2}, A_{e_1e_2:\emptyset} = \overline{A}_{e_1e_2:\emptyset}$, where the barred variables correspond to the situation where orientation of one edge was changed.

$(F_1^{\heartsuit}, F_2^{\heartsuit})$ meets the conditions described for $(F_1^{\diamondsuit}, F_2^{\diamondsuit})$ also. The *trace* of a partition (E_1, E_2) (which is counted in one of A'_{++}, A'_{--}, $2A'_{+-}$, $\ldots$, $2A'''_{+-}$ or $A_{++}, A_{--}, 2A_{+-}$) is defined as $\{C_1 \cap \{x, y, u\}, C_2 \cap \{x, y, u\}\}$, where C_l is the vertex set of a component of E_l containing the vertex a. Table 1 shows contribution of any kind of partition to the numbers appearing in Equation (7). Equation (7) holds since the contributions of partitions of any kind are the same to the left-hand side as to the right-hand side.

Table 1: Contributions of partitions of different traces to the Equation (7).

trace	left-hand side	right-hand side
$\{\{u\},\{u\}\}$	$\Delta A_{--} = 2$	$\Delta A''_{--} = 1$ $\Delta A'''_{--} = 1$
$\{\{u\},\{y\}\}$	$\Delta 2A_{+-} = 1$	$\Delta 2A'''_{+-} = 1$
$\{\{u\},\{y,u\}\}$	$\Delta A_{--} = 1$	$\Delta A''_{--} = 1$
$\{\{u\},\{x\}\}$	$\Delta 2A_{+-} = 1$	$\Delta 2A''_{+-} = 1$
$\{\{u\},\{x,u\}\}$	$\Delta A_{--} = 1$	$\Delta A'''_{--} = 1$
$\{\{u\},\{x,y\}\}$	$\Delta 2A_{+-} = 2$	$\Delta 2A''_{+-} = 1$ $\Delta A'''_{+-} = 1$
$\{\{y\},\{y\}\}$		$\Delta A'_{--} = 1$ $\Delta A'''_{++} = 1$
$\{\{y\},\{y,u\}\}$	$\Delta 2A_{+-} = 1$	$\Delta A'_{--} = 1$
$\{\{y\},\{x\}\}$	$\Delta A_{++} = 1$	$\Delta 2A'_{+-} = 1$
$\{\{y\},\{x,u\}\}$		$\Delta 2A'_{+-} = 1$ $\Delta 2A'''_{+-} = 1$
$\{\{y\},\{x,y\}\}$	$\Delta A_{++} = 1$	$\Delta A'''_{++} = 1$
$\{\{y,u\},\{y,u\}\}$		$\Delta A'_{--} = 1$ $\Delta A''_{--} = 1$
$\{\{y,u\},\{x\}\}$		$\Delta 2A'_{+-} = 1$ $\Delta 2A''_{+-} = 1$
$\{\{y,u\},\{x,u\}\}$	$\Delta A_{--} = 1$	$\Delta 2A'_{+-} = 1$
$\{\{y,u\},\{x,y\}\}$	$\Delta 2A_{+-} = 1$	$\Delta 2A''_{+-} = 1$
$\{\{x\},\{x\}\}$		$\Delta A'_{++} = 1$ $\Delta A''_{++} = 1$
$\{\{x\},\{x,u\}\}$	$\Delta 2A_{+-} = 1$	$\Delta A'_{++} = 1$
$\{\{x\},\{x,y\}\}$	$\Delta A_{++} = 1$	$\Delta A''_{++} = 1$
$\{\{x,u\},\{x,u\}\}$		$\Delta A'_{++} = 1$ $\Delta A'''_{--} = 1$
$\{\{x,u\},\{x,y\}\}$	$\Delta 2A_{+-} = 1$	$\Delta 2A'''_{+-} = 1$
$\{\{x,y\},\{x,y\}\}$	$\Delta A_{++} = 2$	$\Delta A''_{++} = 1$ $\Delta A'''_{++} = 1$

From (5), (6) and (7) we have

$$\begin{aligned}
A_{++} + A_{--} - 2A_{+-} + A_{e_1e_2:\emptyset} - A_{e_1:e_2} = \\
&= A''_{++} + A''_{--} - 2A''_{+-} + A''_{e''_1e''_2:\emptyset} - A''_{e''_1:e''_2} + \\
&\quad + A'''_{++} + A'''_{--} - 2A'''_{+-} + A'''_{e'''_1e'''_2:\emptyset} - A'''_{e'''_1:e'''_2} - \\
&\quad - A'_{++} - A'_{--} + 2A'_{+-} - A'_{e'_1e'_2:\emptyset} + A'_{e'_1:e'_2} = \\
&= 0
\end{aligned}$$

which was to be proven.

□

Theorem 1 can be reformulated as a correlation inequality for spanning trees in a graph. Let $\mathcal{P}$ be the probability distribution of the spanning trees in graph G proportional to their weights, $\mathcal{T}$ the set of all the spanning trees. We have

$$\Pr_{T\sim\mathcal{P}}[T = T_0] = \frac{\|T_0\|}{\|\mathcal{T}\|}$$

for any fixed spanning tree T_0.

Corollary 2. *Let G be a connected graph. For any edges e_1 and e_2, such that e_2 is not a bridge we have*

$$\Pr_{T\sim\mathcal{P}}[e_1 \in T \mid e_2 \notin T] \geq \Pr_{T\sim\mathcal{P}}[e_1 \in T \mid e_2 \in T].$$

Let us note that a similar correlation inequality looks plausible if the spanning trees are replaced by spanning forests. This conjecture was stated by Grimmett and Winkler in [3] and is still open.

Conjecture 3. *Set $\mathcal{F}$ to be the set of all spanning forests of a (weighted) graph G, $\mathcal{B}$ the probability distribution of the spanning forests where the probability of each spanning forest is proportional to its weight. Let e_1 and e_2 be two distinct edges of G. Then*

$$\Pr_{F\sim\mathcal{B}}[e_1 \in F \mid e_2 \notin F] \geq \Pr_{F\sim\mathcal{B}}[e_1 \in F \mid e_2 \in F].$$

The notion of the sets $\mathcal{T}_{e_1,e_2}$, $\mathcal{T}_{e_1,\overline{e_2}}$, $\mathcal{T}_{\overline{e_1},e_2}$ and $\mathcal{T}_{\overline{e_1},\overline{e_2}}$ can be naturally extended to matroids. For a matroid $\mathcal{M} = (E, I)$ with weight $w : E \to \mathbb{R}^+$ on its elements we define

$$\mathcal{T}_{e_1,\overline{e_2}} = \{T \mid T \in I, r(T) = r(\mathcal{M}), e_1 \in T, e_2 \notin T\}$$

and $\mathcal{T}_{e_1,e_2}$, $\mathcal{T}_{\overline{e_1},e_2}$ and $\mathcal{T}_{\overline{e_1},\overline{e_2}}$ similarly. (The two definitions are consistent for graphic matroids.) A matroid is called a *Rayleigh matroid* if it satisfies Equation 1 for any choice of distinct elements $e_1, e_2 \in E$. Graphic matroids are a proper subclass of Rayleigh matroids. See [2, 5] for more details.

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The authors recently learned that Michael LaCroix and David Wagner found independently a very similar proof.

References

[1] Y. Choe, A combinatorial proof of Rayleigh formula for graphs, *preprint*, `http://com2mac.postech.ac.kr/papers/2004/04-30.pdf`.

[2] Y.B. Choe and D.G. Wagner, Rayleigh Matroids, *Combin. Probab. Comput.*, 15(2006), 765-781.

[3] G. Grimmett and S. Winkler, Negative association in uniform forests and connected graphs *Random Structures and Algorithms*, 24 (2004) 444-460.

[4] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchungen der linearen Vertheilung galvanischer Ströme geführt wird, *Ann. Phys. Chem.*, 72 (1847), 497-508.

[5] D. Wagner, Matroid inequalities from electrical network theory, *Electron. J. Combin.*, 11(2) (2005), A1.